STABILITY OF PISTON FLOW MOTION IN A TRAVELING MAGNETIC FIELD

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A formulation is given of the problem of the stability of piston-flow motion in a traveling magnetic field. It is shown that this question reduces to the problem of stability of motion in the presence of constantly acting perturbing forces. The second Lyapunov method is used as the basis to present the sufficient criteria for stability of the flow motion with respect to certain specified quantities.

Piston flow consists of liquid metal slugs (pistons) and the gaseous volumes (plugs) separating them. In reality the pistons may be in the form of either an annulus (cylindrical flow) or a straight bar (plane flow).

In steady-state piston-flow motion the velocities of the external field and centers of inertia of the pistons are equal. We study the stability of this motion for small perturbations of the distance between the pistons.

The system in question (Fig. 1) consists of two plane flows of limited width (the limiting case of an element of cylindrical flows with width equal to the arc along the circumference of the channel). The flows are separated from one another by a wall of thickness Δ and are shifted relative to one another by the distance τ (the pistons are shown hatched). The distance between the centers of inertia of neighboring pistons in each of the flows is 2τ. A direct current flows through the pistons along the y-axis (in the upper pistons the currents are directed into the paper, in the lower they are directed out of the paper). The external magnetic field, created by three-phase alternating current windings (stators) located at the edges of ferromagnetic media bounding the flows from above and below, has the form of a traveling wave of length 2τ and travels in the x direction, as do the flows.

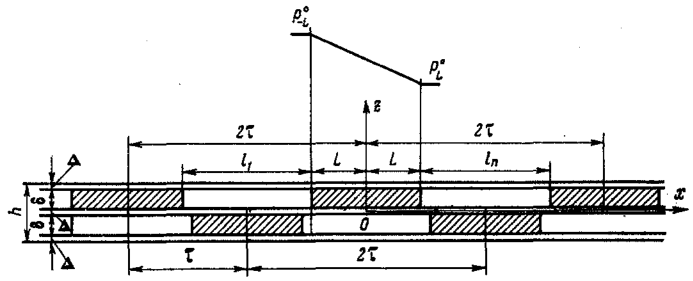

Fig. 1

1. In order to obtain simple dimensionless relations, we assume the common boundaries between the gas and the piston to be plane, and the gas is considered to be an elastic medium. The first assumption is formally valid if the relaxation time of the disturbed motions of the piston is considerably less than the characteristic time for breakdown of the contact surface because of Rayleigh-Taylor instability; the second assumption means that the processes in the gas are adiabatic and is known to be valid for processes such as compression and expansion in a sound wave.

Let us examine the motion of the center piston in Fig. 1. The equation of motion has the form (we use the SI system of units)

$$M\frac{du}{dt} = (P_{-} - P_{+})\,q - F_1 - F_2\,. \tag{1.1}$$

Here M is the mass of a piston of length l (dimension along the y-axis); u is the piston velocity in the laboratory coordinate system x, y, z; P_- and P_+ are, respectively, the pressures to the left and right of the piston; q is the piston area in the yz plane; F_1 is the electromagnetic force; F_2 is the friction force between the piston and the channel wall.

The equation of motion of the gas with account for the assumption on adiabaticity of the processes in the gas in conventional notation has the form

$$\rho \frac{d\mathbf{v}}{dt} = -\operatorname{grad} p, \qquad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \qquad \frac{p}{\rho^{\gamma}} = \text{const} . \tag{1.2}$$

We set

$$\begin{aligned} P_- &= P_-^{\circ} + p_-, \qquad F_1 = F_1^{\circ} + f_1, \qquad u = u^{\circ} + \frac{d\vartheta}{dt} , \\ P_+ &= P_+^{\circ} + p_+, \qquad F_2 = F_2^{\circ} + f_2, \qquad u_g = u^{\circ} + \frac{\partial \theta}{\partial t} . \end{aligned} \tag{1.3}$$

Here we have introduced a simplification by the assumption that the gas motion is one-dimensional ($\mathbf{v} = \mathbf{i}u_g$). The basis for this is the fact that the contribution made by the acoustic pressure p_-, p_+, as shown later, is insignificant. For this same reason we ignore the change of the quantity q in (1.4).

In (1.3) the first terms relate to the steady-state motion of the piston and the second terms are the perturbations of the corresponding quantities; $\vartheta(t)$ is the piston displacement relative to its equilibrium position (by equilibrium we mean the piston position in its steady-state motion). Thus, $\vartheta(t)$ defines the piston position in the X, y, z coordinate system fixed with the piston in its steady-state motion (Fig. 2), $\theta(t, X)$ is the average (in the directed sense) displacement of the gas molecules in the same X, y, z coordinate system.

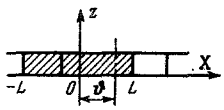

Fig. 2

Substituting (1.3) into (1.1) and (1.2), we obtain

$$M \frac{d^2\vartheta}{dt^2} = (p_- - p_+) q - f_1 - f_2, \tag{1.4}$$

$$\frac{\partial^2 \theta}{\partial t^2} - a_0 \frac{\partial^2 \theta}{\partial X^2} = 0 . \tag{1.5}$$

Here

$$a_0 = \sqrt{\gamma R T_0}, \quad \gamma = \frac{c_p}{c_v}, \quad p_- = -\gamma P_-^{\circ} \left(\frac{\partial \theta}{\partial X}\right)_{-L}, \quad p_+ = -\gamma P_+^{\circ} \left(\frac{\partial \theta}{\partial X}\right)_{L} .$$

Let us evaluate the individual quantities appearing in (1.4).

To determine the electromagnetic force perturbation f_1 we examine the equations of the electrodynamic system (Fig. 1) for small slips of the flows relative to the external magnetic field; here

$$s = \frac{\alpha}{\omega} \frac{d\vartheta}{dt} \quad \left(\alpha = \frac{\pi}{\tau}\right),$$

where ω is the circular frequency of the voltage applied to the stator winding.

We replace the system of two piston flows (Fig. 1) by a single-component flow with the electrical conductivity σ and the current j_v

$$\sigma = \sigma_0 + \sigma_1 \sin \frac{2\pi z}{h} \cos \alpha x, \quad j_v = j_1 \cos \alpha x ,$$

$$\sigma_0 = \frac{\delta}{h} \sigma_p, \quad \sigma_1 = \frac{c_1}{\pi} \left(\cos \frac{2\pi\Delta}{h} + \cos \frac{\pi\Delta}{h}\right), \quad j_1 = \frac{\delta}{h} c_1 j_p .$$

Here σ_p is the electrical conductivity of the piston, j_p is the current density in the piston due to external source, and c_1 is the coefficient of the first term of the Fourier series for the electrical conductivity curve.

The equations of electrodynamics can be written as

$$\Delta A - \mu_0 \sigma \frac{\partial A}{\partial t} = -\mu_0 j_v ,$$

$$\left.\frac{\partial A}{\partial z}\right|_{\pm 1/2 h} = \pm F_{m_1} \sin(\omega s t + \alpha x - \varphi_1) \pm F_{m_2} \sin(\alpha x - \varphi_2) ,$$

$$F_{m_1}=\frac{3}{2}\mu_0 wI_1,\quad F_{m_2}=\frac{3}{2}\mu_0 wI_2,\quad w=\frac{2w_{ph}}{\tau p},$$
$$-U_m\exp[-j^\circ\omega st]=\frac{\partial\psi}{\partial t}+L_s\frac{\partial i}{\partial t}+j^\circ\omega(\psi+L_s i),$$
$$\psi=-j^\circ pwl\int_{-\tau}^{\tau}\left[A\left(\frac{h}{2}\right)+A\left(-\frac{h}{2}\right)\right]\exp(j^\circ\alpha x)\,dx.$$

Here A is the y-component of the vector potential; μ_0 is the magnetic permeability of vacuum; w_{ph} is the number of turns in the stator winding phase; p is the number of stator field waves in the channel length; U_m is the amplitude of the stator voltage; I_1 and I_2 are, respectively, the stator current components due to the voltage $U_m\exp(-j^\circ\omega st)$ and the current in the liquid; φ_1 and φ_2 are the phase angles between I_1 and I_2 for t = 0; L_s is the stator winding leakage inductance; j° is the imaginary unit.

We set

$$A=S-\frac{2\operatorname{ch}{}^1\!/_2\lambda h}{\lambda\operatorname{sh}\lambda h}\operatorname{ch}\lambda zF_{m_1}\exp[-j^\circ(\omega st+\alpha x-\varphi_1)]+$$
$$+\frac{\mu_0 j_1}{\alpha^2}\exp\left[j^\circ\left(\frac{\pi}{2}-\alpha x\right)\right]-\frac{2\operatorname{ch}{}^1\!/_2\alpha h}{\alpha\operatorname{sh}\alpha h}\operatorname{ch}\alpha zF_{m_2}\exp[-j^\circ(\alpha x-\varphi_2)],$$
$$\lambda=r+j^\circ\nu=\alpha\left\{\frac{1}{2}[(1+(\varepsilon s)^2)^{1/2}+1]\right\}^{1/2}-j^\circ\alpha\left\{\frac{1}{2}[(1+(\varepsilon s)^2)^{1/2}-1]\right\}^{1/2}. \quad (1.6)$$

Then

$$\Delta S-\mu_0\sigma\frac{\partial S}{\partial t}=f(t,x,z),$$
$$f(t,x,z)=j^\circ\omega s\mu_0(\sigma-\sigma_0)\frac{2\operatorname{ch}{}^1\!/_2\lambda h}{\lambda\operatorname{sh}\lambda h}\operatorname{ch}\lambda z\,F_{m_1}\exp[-j^\circ(\omega st+\alpha x-\varphi_1)],$$
$$\frac{\partial S}{\partial z}=0\quad\text{for}\quad z=\pm\frac{1}{2}h. \quad (1.7)$$

The quantity S satisfies the periodicity condition with respect to t and x. Therefore, bearing in mind (1.7) and the structure of $f(t,x,z)$, we set

$$S=\sum_0^m a_{m0}\cos\frac{2m\pi z}{h}+\sum_0^{m,n}\cos\frac{2m\pi z}{h}[b_{mn}\cos(\omega st+n\alpha x)+c_{mn}\sin(\omega st+n\alpha x)], \quad (1.8)$$

where a_{m0}, b_{mn}, and c_{mn} are complex numbers. Equation (1.8) satisfies the completeness condition and therefore it must converge in the mean.

Let us utilize the idea of the Galerkin method. It is easy to see that

$$\int_0^{\zeta}\int_{-\tau}^{\tau}\int_{-1/2h}^{1/2h}f(t,x,z)\,\varphi\,dt\,dx\,dz=0\quad\left(\zeta=\frac{2\pi}{\omega s}\right),$$

where φ is any of the coordinate functions from (1.8). Thus, for the coefficients a_{m0}, b_{mn}, c_{mn} we obtain a system of homogeneous algebraic equations. This indicates that these coefficients will be zero. Consequently, S = 0.

The actual magnitude of the vector potential is given by the imaginary part of (1.6), i.e., it is

$$A=F_{m1}[a(z)\sin(\omega st+\alpha x-\varphi_1)-b(z)\cos(\omega st+\alpha x-\varphi_1)]+$$
$$+\frac{2\operatorname{ch}{}^1\!/_2\alpha h}{\alpha\operatorname{sh}\alpha h}\operatorname{ch}\alpha zF_{m_2}\sin(\alpha x-\varphi_2)+\frac{\mu_0 j_1}{\alpha^2}\cos\alpha x, \quad (1.9)$$

where

$$a(z)=q_1\cos\nu z\operatorname{ch}rz-q_2\sin\nu z\operatorname{sh}rz\qquad q_1+j^\circ q_2=\frac{2\operatorname{ch}{}^1\!/_2\lambda h}{\lambda\operatorname{sh}\lambda h}$$
$$b(z)=q_1\sin\nu z\operatorname{sh}rz+q_2\cos\nu z\operatorname{ch}rz,$$

We further set $(\varepsilon s)^2\ll 1$, $h\ll\tau$ and obtain

$$I_1 = \frac{U_m}{x_d}, \quad I_2 = \frac{2pwl\tau\mu_0\omega}{\alpha^2 x_d} j_1, \qquad \varphi_1 = -\delta_c, \quad \varphi_2 = \frac{\pi}{2}, \tag{1.10}$$

where

$$x_d = \omega L_s + \frac{3\cdot 2f}{p} l (2w_\varphi)^2 \mu_0 \frac{1 + \operatorname{ch} \alpha h}{\operatorname{sh} \alpha h}.$$

We find the electromagnetic force acting on the piston from the relation

$$F_1 = \langle f \rangle 2Ll\delta, \quad \langle f \rangle = \frac{1}{2\tau h} \int_{-\tau}^{\tau} \int_{-{}^1\!/_2 h}^{{}^1\!/_2 h} jB_z \, dx dz,$$
$$j = \sigma_0 \frac{\partial A}{\partial t} - j_v, \quad B_z = \frac{\partial A}{\partial x}.$$

Here $\langle f \rangle$ is the average value of the electromagnetic force density.

We have

$$F_1 = F_m \sin \delta_c + C\vartheta + D\frac{d\vartheta}{dt}, \quad F_m = 3\frac{j_1}{\pi}\frac{U_m}{x_d}\mu_0\frac{2w_\varphi}{p}\frac{\delta}{h}Ll,$$
$$C = F_m \alpha \cos \delta_c, \quad D = \left(3w_\varphi \frac{U_m}{x_\alpha}\right)^2 \frac{\mu_0 e\alpha}{\omega (\tau p)^2} \frac{1 + \operatorname{ch} \alpha h}{\operatorname{sh} \alpha h} \frac{\delta}{h} 2Ll. \tag{1.11}$$

In (1.11) the first term corresponds to the steady-state regime. Thus, the perturbation of the electromagnetic force is

$$f_1 = C\vartheta + D\frac{d\vartheta}{dt}. \tag{1.12}$$

The friction force on the wall is

$$F_1 = \zeta \frac{\rho u^2}{2} \frac{L}{h} q,$$

where ζ is the friction coefficient, found from known formulas as a function of the Reynolds number. Hence we obtain for the perturbation of the friction force

$$f_2 = \chi \frac{d\vartheta}{dt}, \qquad \chi = \zeta \rho u^\circ \frac{L}{h} q. \tag{1.13}$$

Thus, for the formulation of (1.4) it remains to find the perturbations p_- and p_+. These quantities depend on the perturbed motion of the gas. Therefore we must examine (1.5).

Among the solutions which formally satisfy (1.5) we take that which will coincide with the solution of (1.4) for $X = \pm L$. Taking (1.12) and (1.13) into account, we can assume that this solution will have either periodic or aperiodic nature for instantaneous perturbations. Therefore we write the solution of (1.5) in one of the following forms:

$$\theta_i = \theta_{1i} + \theta_{2i} = f_{1i}(a_0 t - X - L_i) \exp\left[\frac{k}{a_0}(a_0 t - X - L_i)\right] + f_{2i}(a_0 t + X + L_i) \exp\left[\frac{k}{a_0}(a_0 t + X + L_i)\right] \tag{1.14}$$
$$(i = -, +; \; L_- = -L_+ = L),$$

$$\theta_i = \theta_{1i} + \theta_{2i} = A_{1i} \exp\left[\frac{k_1}{a_0}(a_0 t - X - L_i)\right] + A_{2i} \exp\left[\frac{k_2}{a_0}(a_0 t - X - L_i)\right]$$
$$(A_{1i} = \text{const}, \quad A_{2i} = \text{const}). \tag{1.15}$$

Here and hereafter the subscripts (−) and (+) denote, respectively, the regions to the left and right of the piston.

Let us examine (1.14). It implies that

$$\frac{\partial \theta_{2i}}{\partial X}\bigg|_{-L_i} = \lambda_i \frac{\partial \theta_{1i}}{\partial X}\bigg|_{-L_i},$$

and further

$$\frac{\partial \theta_i}{\partial X}\bigg|_{-L_i} = \frac{1+\lambda_i}{(1-\lambda_i)\,a_0}\,\frac{\partial \theta_i}{\partial t}\bigg|_{-L_i} \tag{1.16}$$

Here λ_i = const and in the general case $\lambda_- \neq \lambda_+$. Since

$$\theta_{i|_{-L_i}} = \vartheta,$$

where ϑ is the piston displacement, we obtain

$$(p_- - p_+)\,q = R\,\frac{d\vartheta}{dt}, \quad R = \left[\frac{1+\lambda_-}{1-\lambda_-}P_-^\circ - \frac{1+\lambda_+}{1-\lambda_+}P_+^\circ\right]\frac{\gamma}{a_0}\,q. \tag{1.17}$$

Thus (1.4) can be written as

$$M\,\frac{d^2\vartheta}{dt^2} - D_e\,\frac{d\vartheta}{dt} + C\vartheta = 0 \quad (D_e = R - D - \chi). \tag{1.18}$$

Hence we obtain

$$\gamma^1 = \frac{D_e}{2M}, \quad \omega = \frac{\sqrt{4MC - D_e^2}}{2M}, \quad 4MC > D_e^2. \tag{1.19}$$

Here γ^1 and ω are the increment and the frequency of the piston oscillations for transient perturbations.

In (1.17) the quantity λ_i is the ratio of the amplitude of the left-traveling wave to the amplitude of the right-traveling wave at the edges of the piston being examined. Consequently the quantity λ_i must depend on the perturbed motion of the neighboring (on the left and right) pistons. Two cases are formally possible: the neighboring pistons are stationary, i. e., they have no displacement relative to the steady-state motion; the neighboring pistons perform the same motion as that being examined but with a different amplitude. The second case is of interest. For this case, we find from the conditions that the oscillations of the neighboring pistons match the sonic oscillation of the gas

$$\lambda_i = \frac{\beta_i^2 - 1}{\exp(\pm 2\gamma^1 l_i / a_0) - 2\beta_i \cos \omega l_i / a_0 \exp(\pm \gamma^1 l_i / a_0) + \beta_i^2}$$

$$\beta_i \,\mathrm{ch}\,\frac{\gamma^1 l_i}{a_0} - \cos\frac{\omega l_i}{a_0} = 0. \tag{1.20}$$

Here the upper sign applies to i = +, the lower sign applies to i = −. Thus, for given distances l_- and l_+ between the pistons (Fig. 1) we can obtain from (1.17), (1.19), and (1.20) all the necessary quantities characterizing the oscillatory motions of the piston for transient perturbations. Numerical calculations of (1.17), (1.19), and (1.20) showed that the contribution of the acoustic pressure $R\,d\vartheta/dt$ is small. For example, for $C = 3.98 \cdot 10^5$ J · m^{-2}, $D_e = 2 \cdot 10^3$ J · · sec · m^{-2}, $P_-^\circ = 250 \cdot 10^5$ N · m^{-2}, $P_+^\circ = 238 \cdot 10^5$ N · m^{-2}, M = 91 kg, a_0 = 440 m · sec^{-1}, $l_- = l_+$ = 0.75 m, q = 0.0375 m^2 we have R = 40 kg · sec · m^{-1}. Thus the acoustic pressure must play a secondary role.

Aperiodic motions (1.15) of the piston are possible if $D_e^2 > 4MC$. In this case neighboring pistons perform the same motions and the value of R is found from the formula

$$R = (P_-^\circ - P_+^\circ)\,\frac{\gamma}{a_0}\,q.$$

Simple estimates show that the role of the quantity R in this case is more prominent than for the case of piston oscillations.

In (1.18) forces having the nature of constantly acting perturbations are not taken into account. Let us examine the basic facts with which the existence of such forces is associated.

The pistons which comprise the flow are coupled with one another. This coupling is provided by the gaseous layers located between the pistons (acoustic pressure on the pistons) and by the electromagnetic field.

As shown previously, the acoustic pressure depends on the static pressures P_-° and P_+° acting on the pistons in the steady-state flow motion, and in the case of oscillatory motions also on the distances l_- and l_+ between the pistons. To ensure normal operation of the device the channel expands slightly along the length. Therefore, in the process of piston motion along the channel these quantities change continuously. However, this change cannot have a stationary nature. This is associated with the fact that the piston dimension along x is still significant (on the order of the length of the stator field half-wave), while the piston dwell time in the channel is short (on the order of hundredths of a second) and may be less than the relaxation time of the mechanical transitional processes (for the numerical values presented above this time is on the order of 0.1 sec). Thus the change of the static gas pressure, associated with useful piston work, must also be accompanied by the formation of acoustic waves and pressures. It is clear that this pressure will have the nature of constantly acting perturbations and will be a function of t, ϑ, and $d\vartheta/dt$.

The piston magnetic mutual-induction field depends on the piston dimensions and distances between them. As the piston travels along the channel, as a result of channel expansion these quantities change continuously, which leads to a change of the mutual-induction field. Moreover, the changes of this field and of the piston self-induction field are associated with the finite channel length. Thus, among the forces acting on the piston there must appear additional electromagnetic forces which have the nature of constantly acting perturbations which depend on t, ϑ, and $d\vartheta/dt$.

To account for the constantly acting perturbations of both acoustic and electromagnetic origin, we must introduce into (1.18) the term $f(t, \vartheta, d\vartheta/dt)$ and write

$$M\frac{d^2\vartheta}{dt^2} - D_e\frac{d\vartheta}{dt} + C\vartheta = f\left(t, \vartheta, \frac{d\vartheta}{dt}\right). \tag{1.21}$$

Analytic description of the function $f(t, \vartheta, d\vartheta/dt)$ is hardly possible. However this situation is not of significant importance, since in the following we need know only the upper limits of this function. In this formulation (1.21) is valid for any of the pistons; therefore we term it the equation of the perturbed piston-flow motion.

2. Let us examine the problem of the stability of the piston-flow motion.

We replace (1.21) by a normal system of differential equations

$$\frac{dy_1}{dt} = a_{11}y_1 + a_{12}y_2 + f_1(t, y_1, y_2), \quad \frac{dy_2}{dt} = a_{21}y_1\ , \tag{2.1}$$

where

$$y_1 = \frac{d\vartheta}{dt}, \quad y_2 = \vartheta, \quad a_{11} = \frac{D_э}{M}, \quad a_{12} = -\frac{C}{M}, \quad a_{21} = 1, \quad f_1 = \frac{f}{M}. \tag{2.2}$$

The unperturbed motion ($y_1 = y_2 = 0$) and the perturbed motion, due to the action of the instantaneous perturbations for $t = 0$, are determined from (2.1) for $f_1 = 0$. For $f_1 \neq 0$ and nonzero initial conditions this same system yields the perturbed motion in the presence of instantaneous and constantly acting perturbations.

Let us find the stability criterion for the motion ($y_1 = y_2 = 0$) on the time interval [0, T] (T is the piston stay time in the channel) relative to the quantities

$$\{\bar{y}_{10}, \bar{y}_{20}\}, \quad \bar{f}_1(t), \quad \{\bar{y}_1^{(t)}, \bar{y}_2(t)\}\ , \tag{2.3}$$

which are, respectively, the given upper bounds of the absolute values of the initial perturbations, the perturbing force, and the subsequent perturbations. We use the second Lyapunov method [1–3] to solve this problem.

In accordance with this method we obtain the sufficient stability criteria on the basis of the upper estimate of the Cauchy integral of the perturbed-motion equations (2.1), which requires the construction of a special positive definite function V of y_1, y_2 and study of this function together with (2.1)

We set

$$V = A \exp\gamma(t) = (A_{11}y_1^2 + A_{22}y_2^2 + 2A_{12}y_1y_2)\exp\gamma(t), \quad A_{12} = A_{21}\,, \tag{2.4}$$

where $\gamma(t)$ is a function which is real on the interval $[0, T]$ and we immediately assume that $\dot{\gamma}(t) > 0$. We shall see later that this requirement does not make the other estimates invalid. The quantities A_{11}, A_{22}, A_{12} must satisfy the equation

$$(a_{11}y_1 + a_{12}y_2)\frac{\partial V}{\partial y_1} + a_{21}y_1\frac{\partial V}{\partial y_2} = (u_1y_1^2 + u_2y_2^2)\exp\gamma(t)\,,$$

where u_1 and u_2 are arbitrary negative time functions.

Setting immediately

$$u_1 = -\frac{a_{21}}{a_{12}}u_2$$

we find

$$A_{11} = -\frac{a_{21}u_2}{a_{11}a_{12}}, \quad A_{12} = \frac{u_2}{2a_{12}}, \quad A_{22} = \left(\frac{1}{a_{11}} - \frac{a_{11}}{2a_{12}a_{21}}\right)u_2\,. \tag{2.5}$$

In order that (2.4) be a positive definite function it is sufficient that the Hurwitz conditions be satisfied for $u_2 < 0$,

$$-a_{12}a_{21} > 0, \quad -a_{11} > 0\,.$$

The first condition, reducing to $C > 0$ in accordance with (2.2), is satisfied automatically; the second yields

$$D_e < 0\,. \tag{2.6}$$

By virtue of (2.1) the total derivative of the function V for $f_1 = 0$ is

$$\dot{V} = B\exp\gamma(t) = (B_{11}y_1^2 + B_{22}y_2^2 + 2B_{12}y_1y_2)\exp\gamma(t), \quad B_{12} = B_{21}\,. \tag{2.7}$$

Here

$$B_{11} = \dot{\gamma}A_{11} - \frac{a_{21}}{a_{12}}u_2, \qquad B_{22} = \dot{\gamma}A_{22} + u_2 \qquad B_{12} = \dot{\gamma}A_{12}\,. \tag{2.8}$$

In obtaining (2.7) and (2.8) we assumed that u_2 = const. The derivative $\dot{V}$ will be a negative definite function if

$$B_{11} < 0, \qquad \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} > 0\,.$$

Hence

$$\frac{|D_e|}{M} > \dot{\gamma} > 0, \quad \left(1 - \frac{M\dot{\gamma}}{|D_e|}\right)^2 - \frac{|D_e|\dot{\gamma}}{2C}\left(1 - \frac{M\dot{\gamma}}{2|D_e|}\right) > 0\,. \tag{2.9}$$

If conditions (2.6) and (2.9) are satisfied, then the upper estimates $X_k(0, t, \bar{y}_0)$ of the absolute value $|y_k(t)|$ of the Cauchy integral of (2.1) for $f_1 = 0$,

$$|y_k(t)| \leqslant X_k(0, t, \bar{y}_0) \leqslant \bar{y}_k(t) \qquad (k = 1,2) \tag{2.10}$$

(where $y_k(t)$, $\bar{y}_k(t)$ are, respectively, the Cauchy integrals for the initial conditions y_0 and $y_0 (y_0 < \bar{y}_0)$) can be found from the formula [4]

$$X_k = \left(A(\bar{y}_0)\frac{M_k}{A_n}\right)^{1/2}\exp\varphi(t, 0) \qquad (k = 1,2)\,. \tag{2.11}$$

Here

$$A(\bar{y}_0) = A_{11}\bar{y}_{10}^2 + A_{22}\bar{y}_{20}^2 + 2A_{12}\bar{y}_{10}\bar{y}_{20}\,,$$
$$A_n = A_{11}A_{22} - A_{12}^2, \quad M_1 = A_{22}, \quad M_2 = A_{11}\,,$$

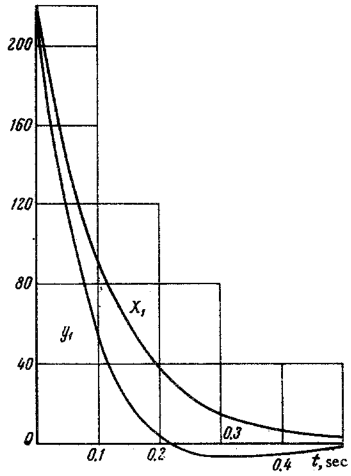

Fig. 3

$$\varphi(t,\ 0) = {}^1/_2\ [\gamma(0) - \gamma(t)]\ . \tag{2.12}$$

Thus we obtain

$$X_1 = \left[\left(\bar{y}_{10}{}^2 + \Delta_1 \bar{y}_{20}{}^2 + \frac{|D_e|}{M}\bar{y}_{10}\bar{y}_{20}\right)\frac{\Delta_1}{\Delta_2}\right]^{-1/2}\exp\left(-\frac{1}{2}\dot{\gamma}_m t\right), \tag{2.13}$$

$$X_2 = \left[\left(\bar{y}_{10}{}^2 + \Delta_1 \bar{y}_{20}{}^2 + \frac{|D_e|}{M}\bar{y}_{10}\bar{y}_{20}\right)\frac{1}{\Delta_2}\right]^{-1/2}\exp\left(-\frac{1}{2}\dot{\gamma}_m t\right),$$

$$\Delta_1 = \frac{D_e^2}{2M^2} + \frac{C}{M}, \quad \Delta_2 = \frac{D_e^2}{4M^2} + \frac{C}{M}\ . \tag{2.14}$$

Here $\mathring{\gamma}_m$ is the maximum value of $\dot{\gamma}$, found from condition (2.10).

So far we have assumed that $f_1 = 0$. For $f_1 \neq 0$, i. e., in the presence of constantly acting perturbing forces, the criteria (2.6) and (2.9) remain valid. The only thing that needs be done is to account for the contribution of f_1 in the formulas for the upper estimates of the Cauchy integrals of (2.10) and write in place of (2.10)

$$Y_k(0,\ t,\ \bar{y}_0) \leqslant \bar{y}_k(t) \quad (k = 1,\ 2)\ , \tag{2.15}$$

where $\bar{y}_k(t)$ is the Cauchy integral of (2.1) for $f_1 \neq 0$ and y_0. The quantity Y_k can be found from the formula [4]

$$Y_k = X_k + \sum_{l=1}^{2}\int_0^t Z_k^{(l)}(t,\tau)\,\bar{f}_l(\tau)\,d\tau \quad (k = 1,\ 2) \tag{2.16}$$

$$Z_k^{(l)}(t,\tau) = z_k^{(l)}(t,\tau)\exp\left(-\frac{1}{2}\dot{\gamma}_m t\right) = \left(A_{ll}\frac{M_k}{A_n}\right)^{1/2}\exp\left(-\frac{1}{2}\dot{\gamma}_m t\right). \tag{2.17}$$

Here X_k are given by (2.13) and (2.14), M_k and A_n are calculated from (2.12). We have

$$Y_1 = X_1 + \frac{2}{\dot{\gamma}_m}\left[1 - \exp\left(-\frac{1}{2}\dot{\gamma}_m t\right)\right] z_1{}^{(1)}\bar{f}_1\ ,$$

$$Y_2 = X_2 + \frac{2}{\dot{\gamma}_m}\left[1 - \exp\left(-\frac{1}{2}\dot{\gamma}_m t\right)\right] z_2{}^{(1)}\bar{f}_1\ ,$$

$$z_1^{(1)} = \Delta_1^{1/2}\Delta_2^{-1/2}, \quad z_2^{(1)} = \Delta_2^{-1/2}. \tag{2.18}$$

Here $\overline{f_1}$ = const is the upper limit of the function $f_1(t, \vartheta, d\vartheta/dt)$.

Thus the sufficient criteria for the stability of piston-flow motion in a traveling magnetic field on the interval [0,T] relative to the prespecified quantities (2.3) are given by the conditions (2.6), (2.9), and (2.15), in which the upper estimates are found from (2.13), (2.14), and (2.18).

The formulas (2.13), (2.14), and (2.18) for the upper estimates of the Cauchy integrals of the perturbed-motion equation for instantaneous perturbations and in the presence of constantly acting perturbing forces contain the same quantities (see (2.11), (2.17). These same quantities define the Lyapunov function V. Therefore, an idea of the quality of these formulas and how successful the function V is constructed can be obtained, for example, by comparing the results given by (2.13) and (2.14) with the exact solution of (2.1) for instantaneous perturbations. This can be done in the problem in question here.

Figure 3 shows the results of calculations of y_1 and its estimates using (2.13) for the following conditions: $D_e = 2\cdot 10^3$ J·sec·m^{-2}, $C = 3.98\cdot 10^5$ J·m^{-2}, $M = 91$ kg, $P_-^{\circ} = 250\cdot 10^5$ N·m^{-2}, $P_+^{\circ} = 238\cdot 10^5$ N·m^{-2}, $a_0 = 440$ m·sec^{-1}, $l_- = l_+ = 0.75$ m, $q = 0.0375$ m^2, $\gamma^1 = -10.34$ sec^{-1}, $\omega = 65.32$ sec^{-1}, $\mathring{\gamma}_m = 18$ sec^{-1}, $\bar{y}_{10}$ = m·sec^{-1}, $\bar{y}_{20} = 0.05$ m. The quantities γ^1 and ω were calculated using (1.17), (1.19) and (1.20), and $\mathring{\gamma}_m$ was determined from conditions (2.9).

This comparison demonstrates the validity of the sufficient criteria obtained above for the stability of piston-flow motion in a traveling magnetic field.

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